Chapter 8

Conservation Laws

Problem 8.1

Example 7.13.

$$\left.\begin{array}{l} \mathbf{E} = \dfrac{\lambda}{2\pi\epsilon_0}\dfrac{1}{s}\,\hat{\mathbf{s}} \\[2ex] \mathbf{B} = \dfrac{\mu_0 I}{2\pi}\dfrac{1}{s}\,\hat{\boldsymbol{\phi}} \end{array}\right\} \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\lambda I}{4\pi^2\epsilon_0}\frac{1}{s^2}\,\hat{\mathbf{z}};$$

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \int_a^b S\,2\pi s\,ds = \frac{\lambda I}{2\pi\epsilon_0}\int_a^b \frac{1}{s}\,ds = \frac{\lambda I}{2\pi\epsilon_0}\ln(b/a).$$

$$\text{But } V = \int_a^b \mathbf{E}\cdot d\mathbf{l} = \frac{\lambda}{2\pi\epsilon_0}\int_a^b \frac{1}{s}\,ds = \frac{\lambda}{2\pi\epsilon_0}\ln(b/a),\ \text{so } \boxed{P = IV.}$$

Problem 7.58.

$$\left.\begin{array}{l} \mathbf{E} = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}} \\[3ex] \mathbf{B} = \mu_0 K\,\hat{\mathbf{x}} = \dfrac{\mu_0 I}{w}\,\hat{\mathbf{x}} \end{array}\right\} \mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{\sigma I}{\epsilon_0 w}\,\hat{\mathbf{y}};$$

$$P = \int \mathbf{S}\cdot d\mathbf{a} = Swh = \frac{\sigma I h}{\epsilon_0},\ \text{but } V = \int \mathbf{E}\cdot d\mathbf{l} = \frac{\sigma}{\epsilon_0}h,\ \text{so } \boxed{P = IV.}$$

Problem 8.2

(a) $\mathbf{E} = \dfrac{\sigma}{\epsilon_0}\,\hat{\mathbf{z}};\ \sigma = \dfrac{Q}{\pi a^2};\ Q(t) = It \ \Rightarrow\ \mathbf{E}(t) = \boxed{\dfrac{It}{\pi\epsilon_0 a^2}\,\hat{\mathbf{z}}.}$

$$B\,2\pi s = \mu_0\epsilon_0\frac{\partial E}{\partial t}\pi s^2 = \mu_0\epsilon_0\frac{I\pi s^2}{\pi\epsilon_0 a^2} \ \Rightarrow\ \mathbf{B}(s,t) = \boxed{\frac{\mu_0 I s}{2\pi a^2}\,\hat{\boldsymbol{\phi}}.}$$

(b) $u_{\text{em}} = \dfrac{1}{2}\left(\epsilon_0 E^2 + \dfrac{1}{\mu_0}B^2\right) = \dfrac{1}{2}\left[\epsilon_0\left(\dfrac{It}{\pi\epsilon_0 a^2}\right)^2 + \dfrac{1}{\mu_0}\left(\dfrac{\mu_0 I s}{2\pi a^2}\right)^2\right] = \boxed{\dfrac{\mu_0 I^2}{2\pi^2 a^4}\left[(ct)^2 + (s/2)^2\right].}$

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(\frac{It}{\pi\epsilon_0 a^2}\right)\left(\frac{\mu_0 I s}{2\pi a^2}\right)(-\hat{\mathbf{s}}) = \boxed{-\frac{I^2 t}{2\pi^2\epsilon_0 a^4}\,s\,\hat{\mathbf{s}}.}$$

$\frac{\partial u_{\text{em}}}{\partial t} = \frac{\mu_0 I^2}{2\pi^2 a^4} 2c^2 t = \frac{I^2 t}{\pi^2 \epsilon_0 a^4}; \quad -\boldsymbol{\nabla} \cdot \mathbf{S} = \frac{I^2 t}{2\pi^2 \epsilon_0 a^4} \boldsymbol{\nabla} \cdot (s\,\hat{\mathbf{s}}) = \frac{I^2 t}{\pi^2 \epsilon_0 a^2} = \frac{\partial u_{\text{em}}}{\partial t}.$ ✓

(c) $U_{\text{em}} = \int u_{\text{em}} w 2\pi s\, ds = 2\pi w \frac{\mu_0 I^2}{2\pi^2 a^4} \int_0^b [(ct)^2 + (s/2)^2] s\, ds = \frac{\mu_0 w I^2}{\pi a^4} \left[(ct)^2 \frac{s^2}{2} + \frac{1}{4}\frac{s^4}{4} \right] \Bigg|_0^b$

$= \boxed{\frac{\mu_0 w I^2 b^2}{2\pi a^4} \left[(ct)^2 + \frac{b^2}{16} \right].}$ Over a surface at radius b: $P_{\text{in}} = -\int \mathbf{S} \cdot d\mathbf{a} = \frac{I^2 t}{2\pi^2 \epsilon_0 a^4} [b\,\hat{\mathbf{s}} \cdot (2\pi b w\,\hat{\mathbf{s}})] = \boxed{\frac{I^2 w t b^2}{\pi \epsilon_0 a^4}.}$

$\frac{dU_{\text{em}}}{dt} = \frac{\mu_0 w I^2 b^2}{2\pi a^4} 2c^2 t = \frac{I^2 w t b^2}{\pi \epsilon_0 a^4} = P_{\text{in}}.$ ✓ (Set $b = a$ for *total.*)

Problem 8.3

$$\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a} - \mu_0 \epsilon_0 \frac{d}{dt} \int \mathbf{S}\, d\tau.$$

The fields are constant, so the second term is zero. The force is clearly in the z direction, so we need

$$\begin{aligned}
(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z &= T_{zx}\, da_x + T_{zy}\, da_y + T_{zz}\, da_z = \frac{1}{\mu_0} \left(B_z B_x\, da_x + B_z B_y\, da_y + B_z B_z\, da_z - \frac{1}{2} B^2\, da_z \right) \\
&= \frac{1}{\mu_0} \left[B_z (\mathbf{B} \cdot d\mathbf{a}) - \frac{1}{2} B^2\, da_z \right].
\end{aligned}$$

Now $\mathbf{B} = \frac{2}{3} \mu_0 \sigma R \omega\, \hat{\mathbf{z}}$ (inside) and $\mathbf{B} = \frac{\mu_0 m}{4\pi r^3} (2\cos\theta\, \hat{\mathbf{r}} + \sin\theta\, \hat{\boldsymbol{\theta}})$ (outside), where $m = \frac{4}{3}\pi R^3 (\sigma \omega R)$. (From Eq. 5.68, Prob. 5.36, and Eq. 5.86.) We want a surface that encloses the entire upper hemisphere—say a hemispherical cap just outside $r = R$ plus the equatorial circular disk.

Hemisphere:

$$\begin{aligned}
B_z &= \frac{\mu_0 m}{4\pi R^3} \left[2\cos\theta\, (\hat{\mathbf{r}})_z + \sin\theta\, (\hat{\boldsymbol{\theta}})_z \right] = \frac{\mu_0 m}{4\pi R^3} \left[2\cos^2\theta - \sin^2\theta \right] = \frac{\mu_0 m}{4\pi R^3} \left(3\cos^2\theta - 1 \right). \\
d\mathbf{a} &= R^2 \sin\theta\, d\theta\, d\phi\, \hat{\mathbf{r}};\ \mathbf{B} \cdot d\mathbf{a} = \frac{\mu_0 m}{4\pi R^3} (2\cos\theta) R^2 \sin\theta\, d\theta\, d\phi;\ da_z = R^2 \sin\theta\, d\theta\, d\phi\, \cos\theta; \\
B^2 &= \left(\frac{\mu_0 m}{4\pi R^3} \right)^2 (4\cos^2\theta + \sin^2\theta) = \left(\frac{\mu_0 m}{4\pi R^3} \right)^2 (3\cos^2\theta + 1). \\
(\overleftrightarrow{\mathbf{T}} \cdot d\mathbf{a})_z &= \frac{1}{\mu_0} \left(\frac{\mu_0 m}{4\pi R^3} \right)^2 \left[(3\cos^2\theta - 1)\, 2\cos\theta R^2 \sin\theta\, d\theta\, d\phi - \frac{1}{2} (3\cos^2\theta + 1)\, R^2 \sin\theta \cos\theta\, d\theta\, d\phi \right] \\
&= \mu_0 \left(\frac{\sigma \omega R}{3} \right)^2 \left[\frac{1}{2} R^2 \sin\theta \cos\theta\, d\theta\, d\phi \right] (12\cos^2\theta - 4 - 3\cos^2\theta - 1) \\
&= \frac{\mu_0}{2} \left(\frac{\sigma \omega R^2}{3} \right)^2 (9\cos^2\theta - 5) \sin\theta \cos\theta\, d\theta\, d\phi. \\
(F_{\text{hemi}})_z &= \frac{\mu_0}{2} \left(\frac{\sigma \omega R^2}{3} \right)^2 2\pi \int_0^{\pi/2} (9\cos^3\theta - 5\cos\theta) \sin\theta\, d\theta = \mu_0 \pi \left(\frac{\sigma \omega R^2}{3} \right)^2 \left[-\frac{9}{4}\cos^4\theta + \frac{5}{2}\cos^2\theta \right] \Bigg|_0^{\pi/2} \\
&= \mu_0 \pi \left(\frac{\sigma \omega R^2}{3} \right)^2 \left(0 + \frac{9}{4} - \frac{5}{2} \right) = -\frac{\mu_0 \pi}{4} \left(\frac{\sigma w R^2}{3} \right)^2.
\end{aligned}$$

Disk:

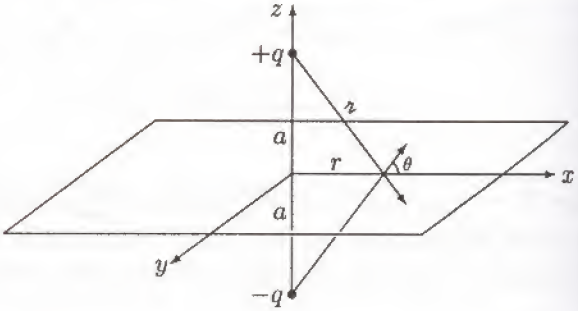
$$\begin{aligned}
B_z &= \frac{2}{3}\mu_0\sigma R\omega; \quad d\mathbf{a} = r\,dr\,d\phi\,\hat{\boldsymbol{\phi}} = -r\,dr\,d\phi\,\hat{\mathbf{z}}; \\
\mathbf{B}\cdot d\mathbf{a} &= -\frac{2}{3}\mu_0\sigma R\omega r\,dr\,d\phi; \quad B^2 = \left(\frac{2}{3}\mu_0\sigma R\omega\right)^2; \quad da_z = -r\,dr\,d\phi. \\
(\overset{\leftrightarrow}{\mathbf{T}}\cdot d\mathbf{a})_z &= \frac{1}{\mu_0}\left(\frac{2}{3}\mu_0\sigma R\omega\right)^2\left[-r\,dr\,d\phi + \frac{1}{2}r\,dr\,d\phi\right] = -\frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0\sigma R\omega\right)^2 r\,dr\,d\phi. \\
(F_{\text{disk}})_z &= -2\mu_0\left(\frac{\sigma\omega R}{3}\right)^2 2\pi\int_0^R r\,dr = -2\pi\mu_0\left(\frac{\sigma\omega R^2}{3}\right)^2.
\end{aligned}$$

Total:

$$\mathbf{F} = -\pi\mu_0\left(\frac{\sigma\omega R^2}{3}\right)^2\left(2+\frac{1}{4}\right)\hat{\mathbf{z}} = \boxed{-\pi\mu_0\left(\frac{\sigma\omega R^2}{2}\right)^2\hat{\mathbf{z}}} \text{ (agrees with Prob. 5.42).}$$

Problem 8.4

(a) $(\overset{\leftrightarrow}{\mathbf{T}}\cdot d\mathbf{a})_z = T_{zx}\,da_x + T_{zy}\,da_y + T_{zz}\,da_z.$



But for the xy plane $da_x = da_y = 0$, and $da_z = -r\,dr\,d\phi$ (I'll calculate the force on the *upper* charge).

$$(\overset{\leftrightarrow}{\mathbf{T}}\cdot d\mathbf{a})_z = \epsilon_0\left(E_zE_z - \frac{1}{2}E^2\right)(-r\,dr\,d\phi).$$

Now $\mathbf{E} = \frac{1}{4\pi\epsilon_0}2\frac{q}{\mathscr{r}^2}\cos\theta\,\hat{\mathbf{r}}$, and $\cos\theta = \frac{r}{\mathscr{r}}$, so $E_z = 0$, $E^2 = \left(\frac{q}{2\pi\epsilon_0}\right)^2\frac{r^2}{(r^2+a^2)^3}$. Therefore

$$\begin{aligned}
F_z &= \frac{1}{2}\epsilon_0\left(\frac{q}{2\pi\epsilon_0}\right)^2 2\pi\int_0^{\in fty}\frac{r^3\,dr}{(r^2+a^2)^3} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\int_0^{\infty}\frac{u\,du}{(u+a^2)^3} \quad (\text{letting } u \equiv r^2) \\
&= \frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\left[-\frac{1}{(u+a^2)} + \frac{a^2}{2\,(u+a^2)^2}\right]\Bigg|_0^{\infty} = \frac{q^2}{4\pi\epsilon_0}\frac{1}{2}\left[0+\frac{1}{a^2}-\frac{a^2}{2a^4}\right] = \boxed{\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}.}\ \checkmark
\end{aligned}$$

(b) In this case $\mathbf{E} = -\frac{1}{4\pi\epsilon_0}2\frac{q}{\mathscr{r}^2}\sin\theta\,\hat{\mathbf{z}}$, and $\sin\theta = \frac{a}{\mathscr{r}}$, so

$$E^2 = E_z^2 = \left(\frac{qa}{2\pi\epsilon_0}\right)^2\frac{1}{(r^2+a^2)^3}, \text{ and hence } (\overset{\leftrightarrow}{\mathbf{T}}\cdot d\mathbf{a})_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2\frac{r\,dr\,d\phi}{(r^2+a^2)^3}. \text{ Therefore}$$

$$F_z = -\frac{\epsilon_0}{2}\left(\frac{qa}{2\pi\epsilon_0}\right)^2 2\pi\int_0^{\infty}\frac{r\,dr}{(r^2+a^2)^3} = -\frac{q^2a^2}{4\pi\epsilon_0}\left[-\frac{1}{4}\frac{1}{(r^2+a^2)^2}\right]_0^{\infty} = -\frac{q^2a^2}{4\pi\epsilon_0}\left[0+\frac{1}{4a^4}\right] = \boxed{-\frac{q^2}{4\pi\epsilon_0}\frac{1}{(2a)^2}.}\ \checkmark$$

Problem 8.5

(a) $E_x = E_y = 0,\ E_z = -\sigma/\epsilon_0$. Therefore

$$T_{xy} = T_{xz} = T_{yz} = \cdots = 0;\quad T_{xx} = T_{yy} = -\frac{\epsilon_0}{2}E^2 = -\frac{\sigma^2}{2\epsilon_0};\quad T_{zz} = \epsilon_0\left(E_z^2 - \frac{1}{2}E^2\right) = \frac{\epsilon_0}{2}E^2 = \frac{\sigma^2}{2\epsilon_0}.$$

$$\boxed{\overleftrightarrow{\mathbf{T}} = \frac{\sigma^2}{2\epsilon_0}\begin{pmatrix}-1&0&0\\0&-1&0\\0&0&+1\end{pmatrix}.}$$

(b) $\mathbf{F} = \oint \overleftrightarrow{\mathbf{T}}\cdot d\mathbf{a}$ ($\mathbf{S} = 0$, since $\mathbf{B} = 0$); integrate over the xy plane: $d\mathbf{a} = -dx\,dy\,\hat{\mathbf{z}}$ (negative because *outward* with respect to a surface enclosing the upper plate). Therefore

$$F_z = \int T_{zz}\,da_z = -\frac{\sigma^2}{2\epsilon_0}A,\ \text{and the force per unit area is } \mathbf{f} = \frac{\mathbf{F}}{A} = \boxed{-\frac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{z}}.}$$

(c) $-T_{zz} = \boxed{\sigma^2/2\epsilon_0}$ is the momentum in the z direction crossing a surface perpendicular to z, per unit area, per unit time (Eq. 8.31).

(d) The recoil force is the momentum delivered per unit time, so the force per unit area on the top plate is

$$\boxed{\mathbf{f} = -\frac{\sigma^2}{2\epsilon_0}\,\hat{\mathbf{z}}}\quad \text{(same as (b))}.$$

Problem 8.6

(a) $\boldsymbol{\wp}_{\text{em}} = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \epsilon_0 EB\,\hat{\mathbf{y}};\quad \mathbf{p}_{\text{em}} = \boxed{\epsilon_0 EBAd\,\hat{\mathbf{y}}.}$

(b) $\mathbf{I} = \int_0^\infty \mathbf{F}\,dt = \int_0^\infty I(\mathbf{l}\times\mathbf{B})\,dt = \int_0^\infty IBd(\hat{\mathbf{z}}\times\hat{\mathbf{x}})\,dt = (Bd\,\hat{\mathbf{y}})\int_0^\infty \left(-\frac{dQ}{dt}\right)dt$ $= -(Bd\hat{\mathbf{y}})[Q(\infty) - Q(0)] = BQd\,\hat{\mathbf{y}}$. But the original field was $E = \sigma/\epsilon_0 = Q/\epsilon_0 A$, so $Q = \epsilon_0 EA$, and hence $\mathbf{I} = \boxed{\epsilon_0 EBAd\,\hat{\mathbf{y}};}$ as expected, the momentum originally stored in the fields (a) is delivered as a kick to the capacitor.

(c) $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} = -\frac{dB}{dt}ld$ (for a length l in the y direction). $-lE(d) + lE(0) = -ld\frac{dB}{dt} \Rightarrow$ $E(d) - E(0) = d\frac{dB}{dt}$. $\mathbf{F} = -\sigma AE(d)\,\hat{\mathbf{y}} + \sigma AE(0)\,\hat{\mathbf{y}} = -\sigma A[E(d) - E(0)]\,\hat{\mathbf{y}} = -\sigma Ad\frac{dB}{dt}\,\hat{\mathbf{y}}$. $\mathbf{I} = \int_0^\infty \mathbf{F}\,dt =$ $-(\sigma Ad\,\hat{\mathbf{y}})\int_0^\infty \frac{dB}{dt}\,dt = -(\sigma Ad\,\hat{\mathbf{y}})[B(\infty) - B(0)] = \sigma AdB\,\hat{\mathbf{y}}$. But $E = \frac{\sigma}{\epsilon_0}$, so $\mathbf{I} = \boxed{\epsilon_0 EBAd\,\hat{\mathbf{y}},}$ as before.

Problem 8.7

$\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}$ (for $a < r < R$; outside the solenoid $B = 0$). The force on a segment dr of spoke is

$$d\mathbf{F} = I'd\mathbf{l}\times\mathbf{B} = I'\mu_0 nI\,dr(\hat{\mathbf{r}}\times\hat{\mathbf{z}}) = -I'\mu_0 nI\,dr\,\hat{\boldsymbol{\phi}}.$$

The torque on the spoke is

$$\mathbf{N} = \int \mathbf{r}\times d\mathbf{F} = I'\mu_0 nI\int_a^R r\,dr(-\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = I'\mu_0 nI\frac{1}{2}\left(R^2 - a^2\right)(-\hat{\mathbf{z}}).$$

Therefore the angular momentum of the cylinders is $\mathbf{L} = \int \mathbf{N}\,dt = -\frac{1}{2}\mu_0 nI(R^2-a^2)\,\hat{\mathbf{z}}\int I'dt$. But $\int I'dt = Q$, so

$$\boxed{\mathbf{L} = -\frac{1}{2}\mu_0 nIQ(R^2-a^2)\,\hat{\mathbf{z}}} \quad \text{(in agreement with Eq. 8.35).}$$

Problem 8.8

(a)

$$\mathbf{E} = \left\{\begin{array}{ll} 0, & (r<R) \\ \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\,\hat{\mathbf{r}}, & (r>R) \end{array}\right\}; \quad \mathbf{B} = \left\{\begin{array}{ll} \frac{2}{3}\mu_0 M\,\hat{\mathbf{z}}, & (r<R) \\ \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right], & (r>R) \end{array}\right\} \text{(Ex. 6.1)}$$

(where $m = \frac{4}{3}\pi R^3 M$); $\wp = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0}{(4\pi)^2}\frac{Qm}{r^5}(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}})\sin\theta$, and $(\hat{\mathbf{r}}\times\hat{\boldsymbol{\theta}}) = \hat{\boldsymbol{\phi}}$, so

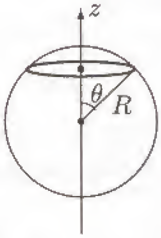
$$\boldsymbol{\ell} = \mathbf{r}\times\wp = \frac{\mu_0}{(4\pi)^2}\frac{mQ}{r^4}\sin\theta(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}).$$

But $(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = -\hat{\boldsymbol{\theta}}$, and only the z component will survive integration, so (since $(\hat{\boldsymbol{\theta}})_z = -\sin\theta$):

$$\mathbf{L} = \frac{\mu_0 mQ}{(4\pi)^2}\,\hat{\mathbf{z}}\int\frac{\sin^2\theta}{r^4}\left(r^2\sin\theta\,dr\,d\theta\,d\phi\right). \quad \int_0^{2\pi} d\phi = 2\pi; \quad \int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}; \quad \int_R^{\infty}\frac{1}{r^2}\,dr = \left(-\frac{1}{r}\right)\Big|_R^{\infty} = \frac{1}{R}.$$

$$\mathbf{L} = \frac{\mu_0 mQ}{(4\pi)^2}\,\hat{\mathbf{z}}\,(2\pi)\left(\frac{4}{3}\right)\left(\frac{1}{R}\right) = \boxed{\frac{2}{9}\mu_0 MQR^2\,\hat{\mathbf{z}}.}$$

(b) Apply Faraday's law to the ring shown:



$$\oint \mathbf{E}\cdot d\mathbf{l} = E(2\pi r\sin\theta) = -\frac{d\Phi}{dt} = -\pi(r\sin\theta)^2\left(\frac{2}{3}\mu_0\frac{dM}{dt}\right)$$

$$\Rightarrow \boxed{\mathbf{E} = -\frac{\mu_0}{3}\frac{dM}{dt}(r\sin\theta)\,\hat{\boldsymbol{\phi}}.}$$

The force on a patch of surface (da) is $d\mathbf{F} = \sigma\mathbf{E}\,da = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}(r\sin\theta)\,da\,\hat{\boldsymbol{\phi}}$ $\left(\sigma = \frac{Q}{4\pi R^2}\right)$.

The torque on the patch is $d\mathbf{N} = \mathbf{r}\times d\mathbf{F} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\left(r^2\sin\theta\right)da\,(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}})$. But $(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}}) = -\hat{\boldsymbol{\theta}}$, and we want only the z component $(\hat{\boldsymbol{\theta}}_z = -\sin\theta)$:

$$\mathbf{N} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\,\hat{\mathbf{z}}\int r^2\sin^2\theta\left(r^2\sin\theta\,d\theta\,d\phi\right).$$

Here $r = R$; $\int_0^{\pi}\sin^3\theta\,d\theta = \frac{4}{3}$; $\int_0^{2\pi}d\phi = 2\pi$, so $\mathbf{N} = -\frac{\mu_0\sigma}{3}\frac{dM}{dt}\,\hat{\mathbf{z}}\,R^4\left(\frac{4}{3}\right)(2\pi) = \boxed{-\frac{2\mu_0}{9}QR^2\frac{dM}{dt}\,\hat{\mathbf{z}}.}$

$$\mathbf{L} = \int\mathbf{N}\,dt = -\frac{2\mu_0}{9}QR^2\,\hat{\mathbf{z}}\int_M^0 dM = \boxed{\frac{2\mu_0}{9}MQR^2\,\hat{\mathbf{z}}} \quad \text{(same as (a)).}$$

(c) Let the charge on the sphere at time t be $q(t)$; the charge *density* is $\sigma = \frac{q(t)}{4\pi R^2}$. The charge below ("south of") the ring in the figure is

$$q_s = \sigma\left(2\pi R^2\right)\int_0^\pi \sin\theta'\,d\theta' = \frac{q}{2}\left(-\cos\theta'\right)\Big|_\theta^\pi = \frac{q}{2}(1+\cos\theta).$$

So the total current crossing the ring (flowing "north") is $I(t) = -\frac{1}{2}\frac{dq}{dt}(1+\cos\theta)$, and hence $\mathbf{K}(t) = \frac{I}{2\pi R\sin\theta}(-\hat{\boldsymbol{\theta}}) = \frac{1}{4\pi R}\frac{dq}{dt}\frac{(1+\cos\theta)}{\sin\theta}\,\hat{\boldsymbol{\theta}}$. The force on a patch of area da is $d\mathbf{F} = (\mathbf{K}\times\mathbf{B})\,da$.

$$\mathbf{B}_{\text{ave}} = \left[\frac{2}{3}\mu_0 M\,\hat{\mathbf{z}} + \frac{\mu_0}{4\pi}\frac{\frac{4}{3}\pi R^3 M}{R^3}\left(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right)\right]\frac{1}{2} = \frac{\mu_0 M}{6}\left[2\,\hat{\mathbf{z}} + 2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}\right];$$

$$\mathbf{K}\times\mathbf{B} = \frac{1}{4\pi R}\frac{dq}{dt}\frac{\mu_0 M}{6}\frac{(1+\cos\theta)}{\sin\theta}\left[2(\hat{\boldsymbol{\theta}}\times\hat{\mathbf{z}}) + 2\cos\theta\,\underbrace{(\hat{\boldsymbol{\theta}}\times\hat{\mathbf{r}})}_{-\hat{\boldsymbol{\phi}}}\right].$$

$$\begin{aligned} d\mathbf{N} &= R\,\hat{\mathbf{r}}\times d\mathbf{F} = \frac{\mu_0 M}{24\pi}\left(\frac{dq}{dt}\right)\frac{(1+\cos\theta)}{\sin\theta}2\Big[\underbrace{\hat{\mathbf{r}}\times(\hat{\boldsymbol{\theta}}\times\hat{\mathbf{z}})}_{\hat{\boldsymbol{\theta}}(\hat{\mathbf{r}}\cdot\hat{\mathbf{z}})-\hat{\mathbf{z}}(\hat{\mathbf{r}}\cdot\hat{\boldsymbol{\theta}})} - \cos\theta\,\underbrace{(\hat{\mathbf{r}}\times\hat{\boldsymbol{\phi}})}_{-\hat{\boldsymbol{\theta}}}\Big]R^2\sin\theta\,d\theta\,d\phi \\ &= \frac{\mu_0 M}{12\pi}\left(\frac{dq}{dt}\right)(1+\cos\theta)R^2\left[\cos\theta\,\hat{\boldsymbol{\theta}} + \cos\theta\,\hat{\boldsymbol{\theta}}\right]d\theta\,d\phi = \frac{\mu_0 M R^2}{6\pi}\left(\frac{dq}{dt}\right)(1+\cos\theta)\cos\theta\,d\theta\,d\phi\,\hat{\boldsymbol{\theta}}. \end{aligned}$$

The x and y components integrate to zero; $(\hat{\boldsymbol{\theta}})_z = -\sin\theta$, so (using $\int_0^{2\pi} d\phi = 2\pi$):

$$\begin{aligned} N_z &= -\frac{\mu_0 M R^2}{6\pi}\left(\frac{dq}{dt}\right)(2\pi)\int_0^\pi(1+\cos\theta)\cos\theta\sin\theta\,d\theta = -\frac{\mu_0 M R^2}{3}\left(\frac{dq}{dt}\right)\left(\frac{\sin^2\theta}{2} - \frac{\cos^3\theta}{3}\right)\Big|_0^\pi \\ &= -\frac{\mu_0 M R^2}{3}\left(\frac{dq}{dt}\right)\left(\frac{2}{3}\right) = -\frac{2\mu_0}{9}MR^2\frac{dq}{dt}. \quad \boxed{\mathbf{N} = -\frac{2\mu_0}{9}MR^2\frac{dq}{dt}\,\hat{\mathbf{z}}.} \end{aligned}$$

Therefore

$$\mathbf{L} = \int \mathbf{N}\,dt = -\frac{2\mu_0}{9}MR^2\,\hat{\mathbf{z}}\int_Q^0 dq = \boxed{\frac{2\mu_0}{9}MR^2 Q\,\hat{\mathbf{z}}} \text{ (same as (a)).}$$

(I used the *average* field at the discontinuity—which is the correct thing to do—but in this case you'd get the same answer using either the inside field or the outside field.)

Problem 8.9

(a) $\mathcal{E} = -\frac{d\Phi}{dt}$; $\Phi = \pi a^2 B$; $B = \mu_0 n I_s$; $\mathcal{E} = I_r R$. So $\boxed{I_r = -\frac{1}{R}\left(\mu_0\pi a^2 n\right)\frac{dI_s}{dt}.}$

(b) $\oint \mathbf{E}\cdot d\mathbf{l} = -\frac{d\Phi}{dt} \Rightarrow E(2\pi a) = -\mu_0\pi a^2 n\frac{dI_s}{dt} \Rightarrow \mathbf{E} = -\frac{1}{2}\mu_0 a n\frac{dI_s}{dt}\,\hat{\boldsymbol{\phi}}$. $\mathbf{B} = \frac{\mu_0 I_r}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\,\hat{\mathbf{z}}$ (Eq. 5.38).

$$\mathbf{S} = \frac{1}{\mu_0}(\mathbf{E}\times\mathbf{B}) = \frac{1}{\mu_0}\left(-\frac{\mu_0 a n}{2}\frac{dI_s}{dt}\right)\left(\frac{\mu_0 I_r}{2}\frac{b^2}{(b^2+z^2)^{3/2}}\right)(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}}) = \boxed{-\frac{1}{4}\mu_0 I_r\frac{dI_s}{dt}\frac{ab^2 n}{(b^2+z^2)^{3/2}}\,\hat{\mathbf{r}}.}$$

Power:

$$P = \int \mathbf{S}\cdot d\mathbf{a} = \int_{-\infty}^{\infty} (S)(2\pi a)\,dz = -\frac{1}{2}\pi\mu_0 a^2 b^2 n I_n \frac{dI_s}{dt}\int_{-\infty}^{\infty} \frac{1}{(b^2+z^2)^{3/2}}\,dx$$

The integral is $\displaystyle \left.\frac{z}{b^2\sqrt{z^2+b^2}}\right|_{-\infty}^{\infty} = \frac{1}{b^2} - \left(-\frac{1}{b^2}\right) = \frac{2}{b^2}$.

$$= -\left(\pi\mu_0 a^2 n \frac{dI_s}{dt}\right) I_r = (RI_r)I_r = {I_r}^2 R. \quad \text{qed}$$

Problem 8.10

According to Eqs. 3.104, 4.14, 5.87, and 6.16, the fields are

$$\mathbf{E} = \left\{\begin{array}{ll} -\dfrac{1}{3\epsilon_0}\mathbf{P}, & (r<R), \\[2ex] \dfrac{1}{4\pi\epsilon_0}\dfrac{1}{r^3}[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}], & (r>R), \end{array}\right\} \quad \mathbf{B} = \left\{\begin{array}{ll} \dfrac{2}{3}\mu_0\mathbf{M}, & (r<R), \\[2ex] \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}\,[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}]\,, & (r>R), \end{array}\right\}$$

where $\mathbf{p} = (4/3)\pi R^3\mathbf{P}$, and $\mathbf{m} = (4/3)\pi R^3\mathbf{M}$. Now $\mathbf{p} = \epsilon_0\int(\mathbf{E}\times\mathbf{B})\,d\tau$, and there are two contributions, one from inside the sphere and one from outside.

Inside:

$$\mathbf{p}_{\text{in}} = \epsilon_0\int\left(-\frac{1}{3\epsilon_0}\mathbf{P}\right)\times\left(\frac{2}{3}\mu_0\mathbf{M}\right)\,d\tau = -\frac{2}{9}\mu_0(\mathbf{P}\times\mathbf{M})\int d\tau = -\frac{2}{9}\mu_0(\mathbf{P}\times\mathbf{M})\frac{4}{3}\pi R^3 = \frac{8}{27}\mu_0\pi R^3(\mathbf{M}\times\mathbf{P}).$$

Outside:

$$\mathbf{p}_{\text{out}} = \epsilon_0\frac{1}{4\pi\epsilon_0}\frac{\mu_0}{4\pi}\int\frac{1}{r^6}\left\{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}]\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}]\right\}\,d\tau.$$

Now $\hat{\mathbf{r}}\times(\mathbf{p}\times\mathbf{m}) = \mathbf{p}(\hat{\mathbf{r}}\cdot\mathbf{m}) - \mathbf{m}(\hat{\mathbf{r}}\cdot\mathbf{p})$, so $\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times(\mathbf{p}\times\mathbf{m})] = (\hat{\mathbf{r}}\cdot\mathbf{m})(\hat{\mathbf{r}}\times\mathbf{p}) - (\hat{\mathbf{r}}\cdot\mathbf{p})(\hat{\mathbf{r}}\times\mathbf{m})$, whereas using the BAC-CAB rule directly gives $\hat{\mathbf{r}}\times[\hat{\mathbf{r}}\times(\mathbf{p}\times\mathbf{m})] = \hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})] - (\mathbf{p}\times\mathbf{m})(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})$. So $\{[3(\mathbf{p}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{p}]\times[3(\mathbf{m}\cdot\hat{\mathbf{r}})\,\hat{\mathbf{r}} - \mathbf{m}]\} = -3(\mathbf{p}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\times\mathbf{m}) + 3(\mathbf{m}\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\times\mathbf{p}) + (\mathbf{p}\times\mathbf{m}) = 3\{\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})] - (\mathbf{p}\times\mathbf{m})\} + (\mathbf{p}\times\mathbf{m}) = -2(\mathbf{p}\times\mathbf{m}) + 3\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})]$.

$$\mathbf{p}_{\text{out}} = \frac{\mu_0}{16\pi^2}\int\frac{1}{r^6}\left\{-2(\mathbf{p}\times\mathbf{m}) + 3\,\hat{\mathbf{r}}[\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m})]\right\}r^2\sin\theta\,dr\,d\theta\,d\phi.$$

To evaluate the integral, set the z axis along $(\mathbf{p}\times\mathbf{m})$; then $\hat{\mathbf{r}}\cdot(\mathbf{p}\times\mathbf{m}) = |\mathbf{p}\times\mathbf{m}|\cos\theta$. Meanwhile, $\hat{\mathbf{r}} = \sin\theta\cos\phi\,\hat{\mathbf{x}} + \sin\theta\sin\phi\,\hat{\mathbf{y}} + \cos\theta\,\hat{\mathbf{z}}$. But $\sin\phi$ and $\cos\phi$ integrate to zero, so the $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms drop out, leaving

$$\begin{aligned}
\mathbf{p}_{\text{out}} &= \frac{\mu_0}{16\pi^2}\left(\int_0^\infty\frac{1}{r^4}\,dr\right)\left\{-2(\mathbf{p}\times\mathbf{m})\int\sin\theta\,d\theta\,d\phi + 3|\mathbf{p}\times\mathbf{m}|\,\hat{\mathbf{z}}\int\cos^2\theta\sin\theta\,d\theta\,d\phi\right\} \\
&= \frac{\mu_0}{16\pi^2}\left.\left(-\frac{1}{3r^3}\right)\right|_R^\infty\left[-2(\mathbf{p}\times\mathbf{m})4\pi + 3(\mathbf{p}\times\mathbf{m})\frac{4\pi}{3}\right] = -\frac{\mu_0}{12\pi R^3}(\mathbf{p}\times\mathbf{m}) \\
&= -\frac{\mu_0}{12\pi R^3}\left(\frac{4}{3}\pi R^3\mathbf{P}\right)\times\left(\frac{4}{3}\pi R^3\mathbf{M}\right) = \frac{4\mu_0}{27}R^3(\mathbf{M}\times\mathbf{P}). \\
\mathbf{p}_{\text{tot}} &= \left(\frac{8}{27} + \frac{4}{27}\right)\mu_0 R^3(\mathbf{M}\times\mathbf{P}) = \boxed{\frac{4}{9}\mu_0 R^3(\mathbf{M}\times\mathbf{P}).}
\end{aligned}$$

Problem 8.11

(a) From Eq. 5.68 and Prob. 5.36,

$$\begin{cases} r < R: \ \mathbf{E} = 0, \ \mathbf{B} = \dfrac{2}{3}\mu_0 \sigma R \omega \, \hat{\mathbf{z}}, \quad \text{with } \sigma = \dfrac{e}{4\pi R^2}; \\ r > R: \ \mathbf{E} = \dfrac{1}{4\pi\epsilon_0}\dfrac{e}{r^2}\,\hat{\mathbf{r}}, \ \mathbf{B} = \dfrac{\mu_0}{4\pi}\dfrac{m}{r^3}(2\cos\theta\,\hat{\mathbf{r}} + \sin\theta\,\hat{\boldsymbol{\theta}}), \ \text{with } m = \dfrac{4}{3}\pi\sigma\omega R^4. \end{cases}$$

The energy stored in the electric field is (Ex. 2.8):

$$W_E = \frac{1}{8\pi\epsilon_0}\frac{e^2}{R}.$$

The energy density of the internal magnetic field is:

$$u_B = \frac{1}{2\mu_0}B^2 = \frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0 R\omega \frac{e}{4\pi R^2}\right)^2 = \frac{\mu_0\omega^2 e^2}{72\pi^2 R^2}, \text{ so } W_{B_{\text{in}}} = \frac{\mu_0\omega^2 e^2}{72\pi^2 R^2}\frac{4}{3}\pi R^3 = \frac{\mu_0 e^2\omega^2 R}{54\pi}.$$

The energy density in the external magnetic field is:

$$u_B = \frac{1}{2\mu_0}\frac{\mu_0^2}{16\pi^2}\frac{m^2}{r^6}\left(4\cos^2\theta + \sin^2\theta\right) = \frac{e^2\omega^2 R^4\mu_0}{18(16\pi^2)}\frac{1}{r^6}\left(3\cos^2\theta + 1\right), \text{ so}$$

$$\begin{aligned} W_{B_{\text{out}}} &= \frac{\mu_0 e^2\omega^2 R^4}{(18)(16)\pi^2}\int_R^\infty \frac{1}{r^6}r^2\,dr \int_0^\pi \left(3\cos^2\theta + 1\right)\sin\theta\,d\theta \int_0^{2\pi} d\phi = \frac{\mu_0 e^2\omega^2 R^4}{(18)(16)\pi^2}\left(\frac{1}{3R^3}\right)(4)(2\pi) = \frac{\mu_0 e^2\omega^2 R}{108\pi}. \\ W_B &= W_{B_{\text{in}}} + W_{b_{\text{out}}} = \frac{\mu_0 e^2\omega^2 R}{108\pi}(2+1) = \frac{\mu_0 e^2\omega^2 R}{36\pi}; \ W = W_E + W_B = \boxed{\frac{1}{8\pi\epsilon_0}\frac{e^2}{R} + \frac{\mu_0 e^2\omega^2 R}{36\pi}.} \end{aligned}$$

(b) Same as Prob. 8.8(a), with $Q \to e$ and $m \to \frac{1}{3}e\omega R^2$: $\boxed{\mathbf{L} = \dfrac{\mu_0 e^2\omega R}{18\pi}\,\hat{\mathbf{z}}.}$

(c) $\dfrac{\mu_0 e^2}{18\pi}\omega R = \dfrac{\hbar}{2} \Rightarrow \omega R = \dfrac{9\pi\hbar}{\mu_0 e^2} = \dfrac{(9)(\pi)(1.05\times10^{-34})}{(4\pi\times10^{-7})(1.60\times10^{-19})^2} = \boxed{9.23\times10^{10}\,\text{m/s.}}$

$$\frac{1}{8\pi\epsilon_0}\frac{e^2}{R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = mc^2; \ \left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = 1 + \frac{2}{9}\left(\frac{9.23\times10^{10}}{3\times10^8}\right)^2 = 2.10\times10^4;$$

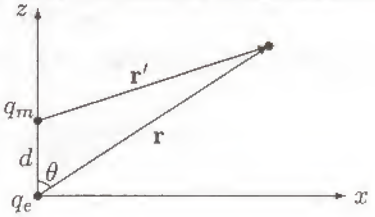
$$R = \frac{(2.01\times10^4)(1.6\times10^{-19})^2}{8\pi(8.85\times10^{-12})(9.11\times10^{-31})(3\times10^8)^2} = \boxed{2.95\times10^{-11}\,\text{m;}} \quad \omega = \frac{9.23\times10^{-10}}{2.95\times10^{-11}} = \boxed{3.13\times10^{21}\,\text{rad/s.}}$$

Since ωR, the speed of a point on the equator, is 300 times the speed of light, this "classical" model is clearly unrealistic.

Problem 8.12

$$\mathbf{E} = \frac{q_e}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3};$$

$$\mathbf{B} = \frac{\mu_0 q_m}{4\pi}\frac{\mathbf{r}'}{r'^3} = \frac{\mu_0 q_m}{4\pi}\frac{(\mathbf{r} - d\,\hat{\mathbf{z}})}{(r^2 + d^2 - 2rd\cos\theta)^{3/2}}.$$



Momentum density (Eq. 8.33):

$$\wp = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \frac{\mu_0 q_e q_m}{(4\pi)^2}\frac{(-d)(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}.$$

Angular momentum density (Eq. 8.34):

$$\boldsymbol{\ell} = (\mathbf{r}\times\wp) = -\frac{\mu_0 q_e q_m d}{(4\pi)^2}\frac{\mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}})}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}.\quad \text{But } \mathbf{r}\times(\mathbf{r}\times\hat{\mathbf{z}}) = \mathbf{r}(\mathbf{r}\cdot\hat{\mathbf{z}}) - r^2\,\hat{\mathbf{z}} = r^2\cos\theta\,\hat{\mathbf{r}} - r^2\,\hat{\mathbf{z}}.$$

The x and y components will integrate to zero; using $(\hat{\mathbf{r}})_z = \cos\theta$, we have:

$$\begin{aligned}\mathbf{L} &= -\frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\hat{\mathbf{z}}\int\frac{r^2(\cos^2\theta-1)}{r^3\left(r^2+d^2-2rd\cos\theta\right)^{3/2}}r^2\sin\theta\,dr\,d\theta\,d\phi.\quad \text{Let } u\equiv\cos\theta:\\ &= \frac{\mu_0 q_e q_m d}{(4\pi)^2}\,\hat{\mathbf{z}}\,(2\pi)\int_{-1}^{1}\int_{0}^{\infty}\frac{r\left(1-u^2\right)}{\left(r^2+d^2-2rdu\right)^{3/2}}\,du\,dr.\end{aligned}$$

Do the r integral first:

$$\int_0^\infty\frac{r\,dr}{\left(r^2+d^2-2rdu\right)^{3/2}} = \left.\frac{(ru-d)}{d(1-u^2)\sqrt{r^2+d^2-2rdu}}\right|_0^\infty = \frac{u}{d\,(1-u^2)} + \frac{d}{d\,(1-u^2)\,d} = \frac{u+1}{d\,(1-u^2)} = \frac{1}{d(1-u)}.$$

Then

$$\mathbf{L} = \frac{\mu_0 q_e q_m d}{8\pi}\,\hat{\mathbf{z}}\,\frac{1}{d}\int_{-1}^{1}\frac{(1-u^2)}{(1-u)}\,du = \frac{\mu_0 q_e q_m}{8\pi}\,\hat{\mathbf{z}}\int_{-1}^{1}(1+u)du = \frac{\mu_0 q_e q_m}{8\pi}\,\hat{\mathbf{z}}\left.\left(u+\frac{u^2}{2}\right)\right|_{-1}^{1} = \boxed{\frac{\mu_0 q_e q_m}{4\pi}\,\hat{\mathbf{z}}.}$$

Problem 8.13

(a) The rotating shell at radius b produces a solenoidal magnetic field:

$$\mathbf{B} = \mu_0 K\,\hat{\mathbf{z}},\ \text{where } K = \sigma_b\omega_b b,\ \text{and } \sigma_b = -\frac{Q}{2\pi bl}.\ \text{So } \mathbf{B} = -\frac{\mu_0\omega_b Q}{2\pi l}\,\hat{\mathbf{z}}\ (a<s<b).$$

The shell at a also produces a magnetic field $(\mu_0\omega_a Q/2\pi l)\,\hat{\mathbf{z}}$, in the region $s<a$, so the total field inside the inner shell is

$$\mathbf{B} = \frac{\mu_0 Q}{2\pi l}\left(\omega_a-\omega_b\right)\hat{\mathbf{z}},\ (s<a).$$

Meanwhile, the electric field is

$$\mathbf{E} = \frac{1}{2\pi\epsilon_0}\frac{\lambda}{s}\,\hat{\mathbf{s}} = \frac{Q}{2\pi\epsilon_0 ls}\,\hat{\mathbf{s}},\quad (a<s<b).$$

$$\wp = \epsilon_0(\mathbf{E}\times\mathbf{B}) = \epsilon_0\left(\frac{Q}{2\pi\epsilon_0 ls}\right)\left(-\frac{\mu_0\omega_b Q}{2\pi l}\right)(\hat{\mathbf{s}}\times\hat{\mathbf{z}}) = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2 s}\,\hat{\boldsymbol{\phi}};\quad \boldsymbol{\ell} = \mathbf{r}\times\wp = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2 s}(\mathbf{r}\times\hat{\boldsymbol{\phi}}).$$

Now $\mathbf{r}\times\hat{\boldsymbol{\phi}} = (s\,\hat{\mathbf{s}}+z\,\hat{\mathbf{z}})\times\hat{\boldsymbol{\phi}} = s\,\hat{\mathbf{z}} - z\,\hat{\mathbf{s}}$, and the $\hat{\mathbf{s}}$ term integrates to zero, so

$$\mathbf{L} = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2}\,\hat{\mathbf{z}}\int d\tau = \frac{\mu_0\omega_b Q^2}{4\pi^2 l^2}\pi(b^2-a^2)l\,\hat{\mathbf{z}} = \boxed{\frac{\mu_0\omega_b Q^2(b^2-a^2)}{4\pi l}\,\hat{\mathbf{z}}.}$$

(b) The extra electric field induced by the changing magnetic field due to the rotating shells is given by $E\,2\pi s = -\frac{d\Phi}{dt} \Rightarrow \mathbf{E} = -\frac{1}{2\pi s}\frac{d\Phi}{dt}\,\hat{\boldsymbol{\phi}}$, and in the region $a < s < b$

$$\Phi = \frac{\mu_0 Q}{2\pi l}(\omega_a - \omega_b)\,\pi a^2 - \frac{\mu_0 Q \omega_b}{2\pi l}\pi\left(s^2 - a^2\right) = \frac{\mu_0 Q}{2l}\left(\omega_a a^2 - \omega_b s^2\right);\ \mathbf{E}(s) = -\frac{1}{2\pi s}\frac{\mu_0 Q}{2l}\left(a^2\frac{d\omega_a}{dt} - s^2\frac{d\omega_b}{dt}\right)\hat{\boldsymbol{\phi}}.$$

In particular,

$$\mathbf{E}(a) = -\frac{\mu_0 Q a}{4\pi l}\left(\frac{d\omega_a}{dt} - \frac{d\omega_b}{dt}\right)\hat{\boldsymbol{\phi}}, \quad \text{and } \mathbf{E}(b) = -\frac{\mu_0 Q}{4\pi l b}\left(a^2\frac{d\omega_a}{dt} - b^2\frac{d\omega_b}{dt}\right)\hat{\boldsymbol{\phi}}.$$

The torque on a shell is $\mathbf{N} = \mathbf{r} \times q\mathbf{E} = qsE\,\hat{\mathbf{z}}$, so

$$\begin{aligned}
\mathbf{N}_a &= Qa\left(-\frac{\mu_0 Q a}{4\pi l}\right)\left(\frac{d\omega_a}{dt} - \frac{d\omega_b}{dt}\right)\hat{\mathbf{z}};\quad \mathbf{L}_a = \int_0^\infty \mathbf{N}_a\,dt = -\frac{\mu_0 Q^2 a^2}{4\pi l}(\omega_a - \omega_b)\,\hat{\mathbf{z}}.\\
\mathbf{N}_b &= -Qb\left(-\frac{\mu_0 Q}{4\pi l b}\right)\left(a^2\frac{d\omega_a}{dt} - b^2\frac{d\omega_b}{dt}\right)\hat{\mathbf{z}};\quad \mathbf{L}_b = \int_0^\infty \mathbf{N}_b\,dt = \frac{\mu_0 Q^2}{4\pi l}\left(a^2\omega_a - b^2\omega_b\right)\hat{\mathbf{z}}.\\
\mathbf{L}_{\text{tot}} &= \mathbf{L}_a + \mathbf{L}_b = \frac{\mu_0 Q^2}{4\pi l}\left(a^2\omega_a - b^2\omega_b - a^2\omega_a + a^2\omega_b\right)\hat{\mathbf{z}} = \boxed{-\frac{\mu_0 Q^2 \omega_b}{4\pi l}(b^2 - a^2)\,\hat{\mathbf{z}}.}
\end{aligned}$$

Thus the reduction in the final mechanical angular momentum (b) is equal to the residual angular momentum in the fields (a). ✓

Problem 8.14

$\mathbf{B} = \mu_0 n I\,\hat{\mathbf{z}},\ (s < R);\quad \mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\boldsymbol{\mathscr{r}}}{\mathscr{r}^3}$, where $\boldsymbol{\mathscr{r}} = (x - a, y, z)$.

$$\wp = \epsilon_0(\mathbf{E} \times \mathbf{B}) = \epsilon_0(\mu_0 n I)\left(\frac{q}{4\pi\epsilon_0}\right)\frac{1}{\mathscr{r}^3}(\boldsymbol{\mathscr{r}} \times \hat{\mathbf{z}}) = \frac{\mu_0 q n I}{4\pi \mathscr{r}^3}[y\,\hat{\mathbf{x}} - (x - a)\,\hat{\mathbf{y}}].$$

Linear Momentum.

$$\begin{aligned}
\mathbf{p} &= \int \wp\,d\tau = \frac{\mu_0 q n I}{4\pi}\int \frac{y\,\hat{\mathbf{x}} - (x-a)\,\hat{\mathbf{y}}}{[(x-a)^2 + y^2 + z^2]^{3/2}}\,dx\,dy\,dz. \text{ The } \hat{\mathbf{x}} \text{ term is odd in } y\text{; it integrates to zero.}\\
&= -\frac{\mu_0 q n I}{4\pi}\,\hat{\mathbf{y}}\int \frac{(x-a)}{[(x-a)^2 + y^2 + z^2]^{3/2}}\,dx\,dy\,dz. \quad \text{Do the } z \text{ integral first :}\\
&\quad \left.\frac{z}{[(x-a)^2 + y^2]\sqrt{(x-a)^2 + y^2 + z^2}}\right|_{-\infty}^{\infty} = \frac{2}{[(x-a)^2 + y^2]}.\\
&= -\frac{\mu_0 q n I}{2\pi}\,\hat{\mathbf{y}}\int \frac{(x-a)}{[(x-a)^2 + y^2]}\,dx\,dy. \quad \text{Switch to polar coordinates :}\\
&\quad x = s\cos\phi,\ y = s\sin\phi,\ dx\,dy \Rightarrow s\,ds\,d\phi;\ [(x-a)^2 + y^2] = s^2 + a^2 - 2sa\cos\phi.\\
&= -\frac{\mu_0 q n I}{2\pi}\,\hat{\mathbf{y}}\int \frac{(s\cos\phi - a)}{(s^2 + a^2 - 2sa\cos\phi)}s\,ds\,d\phi\\
&\quad \text{Now } \int_0^{2\pi}\frac{\cos\phi\,d\phi}{(A + B\cos\phi)} = \frac{2\pi}{B}\left(1 - \frac{A}{\sqrt{A^2 - B^2}}\right);\quad \int_0^{2\pi}\frac{d\phi}{(A + B\cos\phi)} = \frac{2\pi}{\sqrt{A^2 - B^2}}.\\
&\quad \text{Here } A^2 - B^2 = (s^2 + a^2)^2 - 4s^2a^2 = s^4 + 2s^2a^2 + a^4 - 4s^2a^2 = (s^2 - a^2)^2;\ \sqrt{A^2 - B^2} = a^2 - s^2.\\
&= \frac{\mu_0 q n I}{2a}\,\hat{\mathbf{y}}\int\left[1 - \left(\frac{a^2 + s^2}{a^2 - s^2}\right) + \frac{2a^2}{(a^2 - s^2)}\right]s\,ds = \frac{\mu_0 q n I}{a}\,\hat{\mathbf{y}}\int_0^R s\,ds = \boxed{\frac{\mu_0 q n I R^2}{2a}\,\hat{\mathbf{y}}.}
\end{aligned}$$

Angular Momentum.

$$\boldsymbol{\ell} = \mathbf{r}\times\boldsymbol{\wp} = \frac{\mu_0 qnI}{4\pi\mathscr{r}^3}\mathbf{r}\times[y\,\hat{\mathbf{x}} - (x-a)\,\hat{\mathbf{y}}] = \frac{\mu_0 qnI}{4\pi\mathscr{r}^3}\left\{z(x-a)\,\hat{\mathbf{x}} + zy\,\hat{\mathbf{y}} - [x(x-a)+y^2]\,\hat{\mathbf{z}}\right\}.$$

The $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ terms are odd in z, and integrate to zero, so

$$\mathbf{L} = -\frac{\mu_0 qnI}{4\pi}\,\hat{\mathbf{z}}\int\frac{x^2+y^2-xa}{[(x-a)^2+y^2+z^2]^{3/2}}\,dx\,dy\,dz.$$ The z integral is the same as before.

$$= -\frac{\mu_0 qnI}{2\pi}\,\hat{\mathbf{z}}\int\frac{x^2+y^2-xa}{[(x-a)^2+y^2]}\,dx\,dy = -\frac{\mu_0 qnI}{2\pi}\,\hat{\mathbf{z}}\int\frac{s-a\cos\phi}{(s^2+a^2-2sa\cos\phi)}s^2\,ds\,d\phi$$

$$= -\mu_0 qnI\,\hat{\mathbf{z}}\int\left[\frac{s^2}{a^2-s^2}+\left(1-\frac{a^2+s^2}{a^2-s^2}\right)\right]s\,ds = -\mu_0 qnI\,\hat{\mathbf{z}}\int_0^R\frac{s^2-s^2}{a^2-s^2}s\,ds = \boxed{\text{zero.}}$$

Problem 8.15

(a) If we're only interested in the work done on *free* charges and currents, Eq. 8.6 becomes $\frac{dW}{dt} = \int_{\mathcal{V}}(\mathbf{E}\cdot\mathbf{J}_f)\,d\tau$. But $\mathbf{J}_f = \boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$ (Eq. 7.55), so $\mathbf{E}\cdot\mathbf{J}_f = \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H}) - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t}$. From product rule #6, $\boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H})$, while $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$, so $\mathbf{E}\cdot(\boldsymbol{\nabla}\times\mathbf{H}) = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})$. Therefore $\mathbf{E}\cdot\mathbf{J}_f = -\mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t} - \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} - \boldsymbol{\nabla}\cdot(\mathbf{E}\times\mathbf{H})$, and hence

$$\frac{dW}{dt} = -\int_{\mathcal{V}}\left(\mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}\right)d\tau - \oint_{\mathcal{S}}(\mathbf{E}\times\mathbf{H})\cdot d\mathbf{a}.$$

This is Poynting's theorem for the fields in matter. Evidently the Poynting vector, representing the power per unit area transported by the fields, is $\mathbf{S} = \mathbf{E}\times\mathbf{H}$, and the rate of change of the electromagnetic energy density is $\frac{\partial u_{\text{em}}}{\partial t} = \mathbf{E}\cdot\frac{\partial\mathbf{D}}{\partial t} + \mathbf{H}\cdot\frac{\partial\mathbf{B}}{\partial t}$.

For *linear* media, $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \frac{1}{\mu}\mathbf{B}$, with ϵ and μ constant (in time); then

$$\frac{\partial u_{\text{em}}}{\partial t} = \epsilon\mathbf{E}\cdot\frac{\partial\mathbf{E}}{\partial t} + \frac{1}{\mu}\mathbf{B}\cdot\frac{\partial\mathbf{B}}{\partial t} = \frac{1}{2}\epsilon\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{E}) + \frac{1}{2\mu}\frac{\partial}{\partial t}(\mathbf{B}\cdot\mathbf{B}) = \frac{1}{2}\frac{\partial}{\partial t}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H}),$$

so $u_{\text{em}} = \frac{1}{2}(\mathbf{E}\cdot\mathbf{D} + \mathbf{B}\cdot\mathbf{H})$. qed

(b) If we're only interested in the force on *free* charges and currents, Eq. 8.15 becomes $\mathbf{f} = \rho_f\mathbf{E} + \mathbf{J}_f\times\mathbf{B}$. But $\rho_f = \boldsymbol{\nabla}\cdot\mathbf{D}$, and $\mathbf{J}_f = \boldsymbol{\nabla}\times\mathbf{H} - \frac{\partial\mathbf{D}}{\partial t}$, so $\mathbf{f} = \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{D}) + (\boldsymbol{\nabla}\times\mathbf{H})\times\mathbf{B} - \left(\frac{\partial\mathbf{D}}{\partial t}\right)\times\mathbf{B}$. Now $\frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) = \frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} + \mathbf{D}\times\left(\frac{\partial\mathbf{B}}{\partial t}\right)$, and $\frac{\partial\mathbf{B}}{\partial t} = -\boldsymbol{\nabla}\times\mathbf{E}$, so $\frac{\partial\mathbf{D}}{\partial t}\times\mathbf{B} = \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}) + \mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E})$, and hence $\mathbf{f} = \mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{H}) - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B})$. As before, we can with impunity add the term $\mathbf{H}(\boldsymbol{\nabla}\cdot\mathbf{B})$, so

$$\mathbf{f} = \{[\mathbf{E}(\boldsymbol{\nabla}\cdot\mathbf{D}) - \mathbf{D}\times(\boldsymbol{\nabla}\times\mathbf{E})] + [\mathbf{H}(\boldsymbol{\nabla}\cdot\mathbf{B}) - \mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{H})]\} - \frac{\partial}{\partial t}(\mathbf{D}\times\mathbf{B}).$$

The term in curly brackets can be written as the divergence of a stress tensor (as in Eq. 8.21), and the last term is (minus) the rate of change of the momentum density, $\boldsymbol{\wp} = \mathbf{D}\times\mathbf{B}$.